Math 105 Chapter 1: Geometry, vectors, planes.

We begin by reviewing (very breifly) vectors and the geometry of $\mathbb{R}^n$.

Definition: We define n-dimensional euclidean space by $\mathbb{R}^n$.

$$\mathbb{R}^n = \{ (x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R} \}$$

We will be working mainly in $\mathbb{R}^2, \mathbb{R}^3$.

Definition: A point $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ is called a vector.

Note in $\mathbb{R}^2, \mathbb{R}^3$ we call the first, second and third components the "x", "y", "z" components respectively. Visually you can think of an arrow

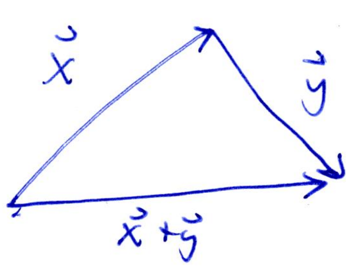
To use vectors, we want to define some operations on them.

- Addition: If $\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, define

$$\vec{x} + \vec{y} = (x_1 + y_1, \dots, x_n + y_n)$$

eg $(2, 3, 4) + (1, -1, \sqrt{2}) = (3, 2, 4+\sqrt{2})$

geometrically: $\vec{x} + \vec{y}$ is the resulting vector when you join the vectors head to tail.

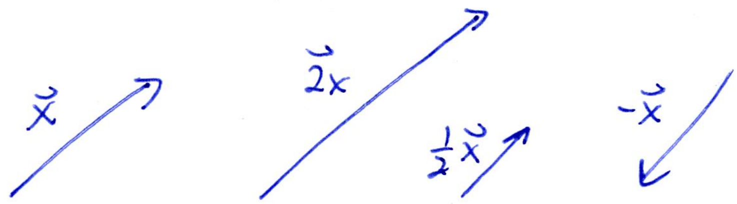


• Stretching: Given $\vec{x} = (x_1, \dots, x_n)$, $c \in \mathbb{R}$, define

$$c\vec{x} = (cx_1, \dots, cx_n)$$

eg $\frac{1}{3}(1,2,3) = (\frac{1}{3}, \frac{2}{3}, 1)$

geometrically: Your stretching the vector by c. Note negative c means you change direction.



Definition: We define the legnth or magnitude or norm of a vector $\vec{x} = (x_1, \dots, x_n)$ by:

$$\|\vec{x}\| = \sqrt{x_1^2 + \dots + x_n^2}$$

Properties of $\|\cdot\|$:

① $\|\vec{x}\| \geq 0$, $\|\vec{x}\| = 0$ if and only if $\vec{x} = 0$

② $\|c\vec{x}\| = |c|\|\vec{x}\|$, $c \in \mathbb{R}$

③ $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

①, ② follow from the definition, ③ is a bit more work, you will be asked to show on the first assignment.

Definition: If $\vec{x}$ is a vector then the direction of $\vec{x}$ is

$$\hat{x} = \frac{\vec{x}}{\|\vec{x}\|}$$

Definition: If $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_n)$ are vectors, we define the dot product of $\vec{x}, \vec{y}$ to be

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n$$

eg: $(1, \sqrt{2}, 3) \cdot (10, -1, -2) = (1)(10) + (\sqrt{2})(-1) + (3)(-2)$
$= 4 - \sqrt{2}$

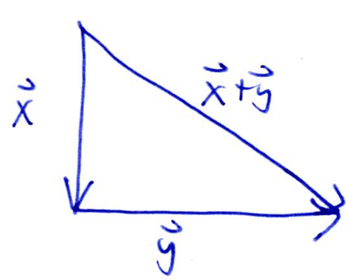
We will see the geometric significance soon.

Note: $\vec{x} \cdot \vec{x} = x_1 x_1 + \cdots + x_n x_n = x_1^2 + \cdots + x_n^2 = \|\vec{x}\|^2$

Properties of dot product:

1. $(\vec{x} \cdot \vec{y}) \cdot \vec{v} = \vec{x} \cdot (\vec{y} \cdot \vec{v})$ (associativity)
2. $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$ (commutativity)
3. $\vec{v} \cdot (c\vec{x} + \vec{y}) = c\vec{v} \cdot \vec{x} + \vec{v} \cdot \vec{y}$ (linearity)
4. $\vec{x} \cdot \vec{x} = \|\vec{x}\|^2$

Recall: In $\mathbb{R}^2$ $\vec{x}, \vec{y}$ are perpindicular if and only if they satisfy the pythagorean theorem; Ie.



$$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

What is $\|\vec{x}+\vec{y}\|^2$?

$$\begin{aligned}\|\vec{x}+\vec{y}\|^2 &= (\vec{x}+\vec{y})\bullet(\vec{x}+\vec{y})\\ &= \vec{x}\bullet\vec{x}+\vec{x}\bullet\vec{y}+\vec{y}\bullet\vec{x}+\vec{y}\bullet\vec{y}\\ &= \|\vec{x}\|^2+2\,\vec{x}\bullet\vec{y}+\|\vec{y}\|^2\end{aligned}$$

So this motivates the following definition:

Definition: $\vec{x}, \vec{y}$ are said to be perpindicular / orthogonal / normal if

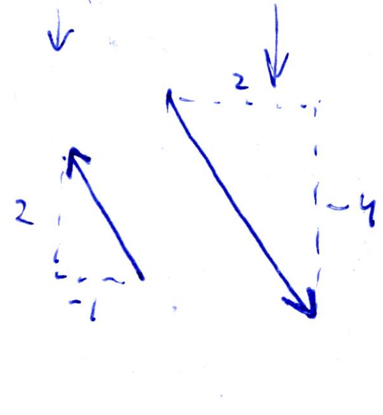
$$\vec{x}\bullet\vec{y}=0$$

$\vec{x}, \vec{y} \neq 0$ are said to be parallel if there is a $c \in \mathbb{R}$ such that

$$\vec{x} = c\,\vec{y}$$

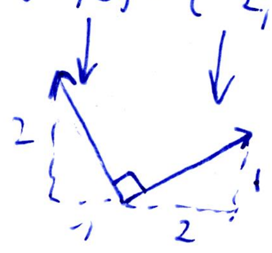
eg: • In $\mathbb{R}^2$, $(-1,2)$, $(2,-6)$ are parallel since

$$(-1,2) = \frac{-1}{2}(2,-4)$$



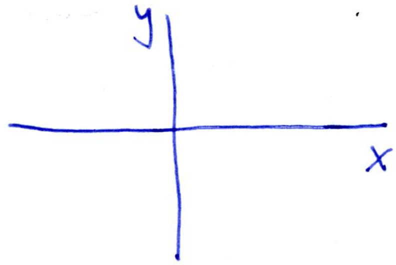
• In $\mathbb{R}^2$, $(-1,2)$, $(2,1)$ are perpindicular since

$$(-1,2)\bullet(2,1) = (-1)(2)+(2)(1) = 0$$

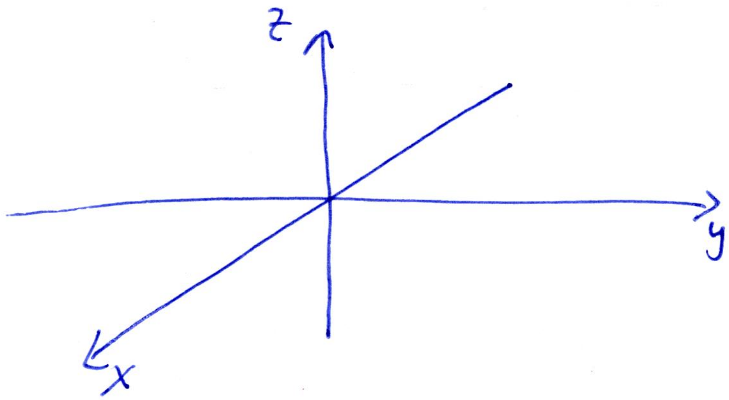


Remark: now on in this course we will be working in $\mathbb{R}^3$.

Note in $\mathbb{R}^2$ we often use the "x,y-axis" shown below:



To truly draw a function in $\mathbb{R}^3$ we would need a "z-axis" to come into or out of the page. Since unfortunately we all don't have 3D-pens and printers we will try to use the following visual manipulation.



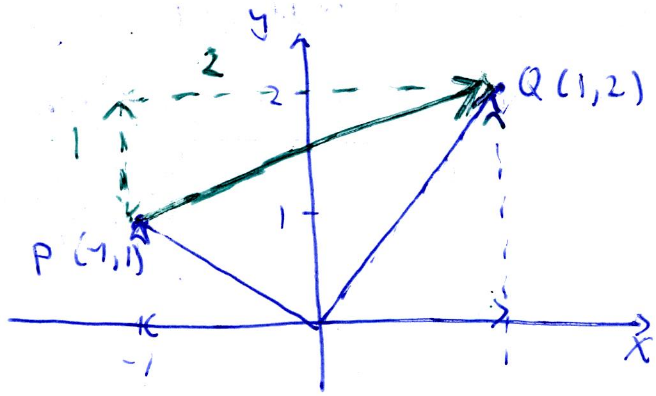
You will need to get used this.

Recall: In math 104, you developed the concept of a tangent line of a function $f(x)$ through the point $x=a$. Once we begin talking about functions of multiple variables we will want a higher dimensional analog of the tangent line. For a function of n-variables we want an n-dimensional analog of a line, called a "plane".

Definition: Given 2 points $\vec{P} = (x_1, \dots, x_n)$, $\vec{Q} = (y_1, \dots, y_n) \in \mathbb{R}^n$, then the vector from P to Q is given by

$$\overrightarrow{QP} = (y_1 - x_1, \dots, y_n - x_n)$$
$$= \vec{Q} - \vec{P}$$

eg $\vec{P} = (-1, 1)$, $\vec{Q} = (1, 2)$, $\overrightarrow{QP} = (1, 2) - (-1, 1) = (2, 1)$



Note: $\overrightarrow{QP} = -\overrightarrow{PQ}$

Definition: Given a vector $0 \neq \vec{n} \in \mathbb{R}^3$ and a point $\vec{X}_0 \in \mathbb{R}^3$ then the set of points $\vec{X} \in \mathbb{R}^3$ whose direction from $\vec{X}_0$ is perpindicular to $\vec{n}$ is called a plane.

This is a very abstract definition, so what does this mean?

Well if $\vec{X} = (x, y, z)$, $\vec{X}_0 = (x_0, y_0, z_0)$ then the direction from $\vec{X}_0$ to $\vec{x}$ is

$$\overrightarrow{XX_0} = \vec{X} - \vec{X}_0$$
$$= (x - x_0, y - y_0, z - z_0)$$

We also have $\overrightarrow{XX_0}$ is perpindicular to $\vec{n} = (a, b, c)$.

So this gives us the following equation:

$$\vec{n} \cdot \overrightarrow{XX_0} = 0$$

$$\Rightarrow (a, b, c) \cdot (X - X_0, y - y_0, z - z_0) = 0$$

$$\Rightarrow \boxed{a(X - X_0) + b(y - y_0) + c(z - z_0) = 0}$$

The last line is known as the <u>equation of a plane</u>.

We can further simplify

$$\vec{n} \cdot \overrightarrow{XX_0} = 0$$

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

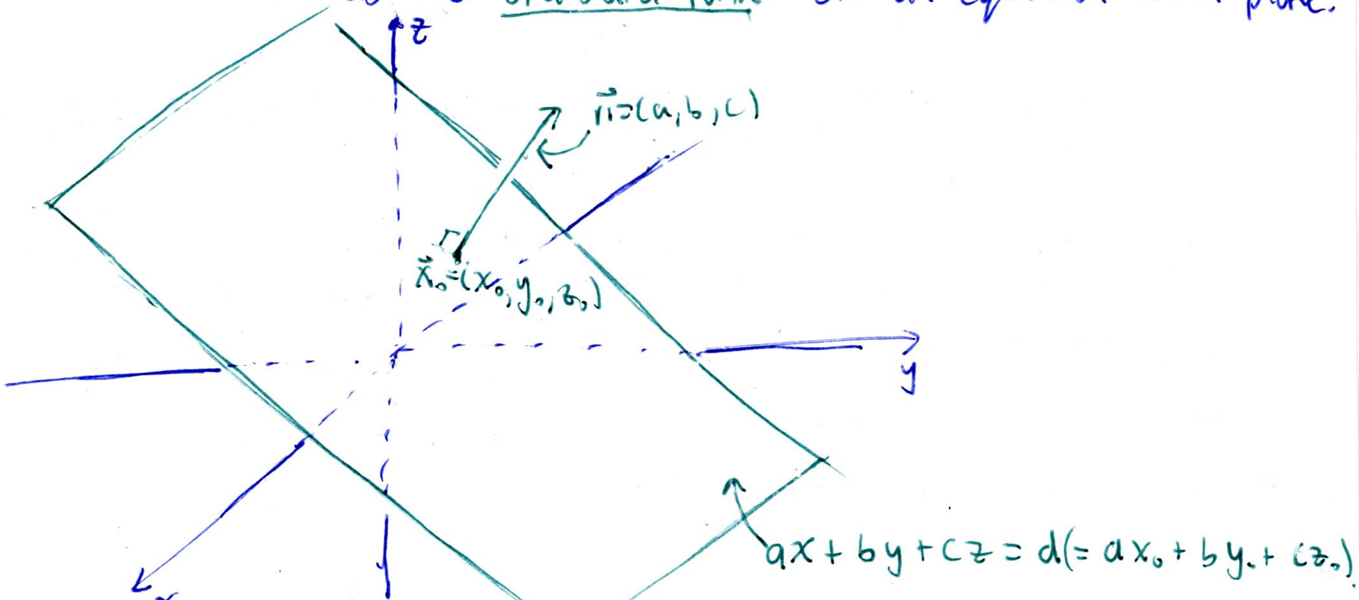
$$\vec{n} \cdot \vec{x} - \vec{n} \cdot \vec{x}_0 = 0$$

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$$

$$(a, b, c) \cdot (x, y, z) = \vec{n} \cdot \vec{x}_0$$

$$\boxed{ax + by + cz = d}, \quad d = \vec{n} \cdot \vec{x}_0 = ax_0 + by_0 + cz_0$$

The last line is known as the <u>standard form</u> of an equation of a plane.



Just like vectors, we can define what it means for two planes to be perpindicular and parallel.

Definition: Let P_1, P_2 be planes with normal vectors $\vec{n}_1, \vec{n}_2 \neq 0$ respectively. Then we say P_1 is orthogonal or perpindicular to P_2 if $\vec{n}_1, \vec{n}_2$ are perpindicular. Similarly P_1 is parallel to P_2 if $\vec{n}_1, \vec{n}_2$ are parallel.

eg. $x + y = 1$ is perpindicular to $z = 2$ since

$$(1,1,0) \cdot (0,0,1) = 0$$

$x + 2y - z = 1$ is parallel to $2x + 4y - 2z = \pi$ since

$$(1,2,-1) = \tfrac{1}{2}(2,4,-2)$$

Note that to determine if planes P_1, P_2 with equations

$$P_1 : a_1 x + b_1 y + c_1 z = d_1$$
$$P_2 : a_2 x + b_2 y + c_2 z = d_2$$

are parallel/orthogonal, we only care out the normal vectors.

$$(a_1, b_1, c_1), \ (a_2, b_2, c_2).$$

In particular d_1, d_2 don't matter. Recall d_1, d_2 are determined by the points that the planes pass through, so this implies to determine if 2 planes are parallel/perpindicular the points that the planes cross do not matter!

Notation: We often use $\vec{i}, \vec{j}, \vec{k}$ to refer to the following common vectors in $\mathbb{R}^3$:

$$\vec{i} = (1,0,0)$$
$$\vec{j} = (0,1,0)$$
$$\vec{k} = (0,0,1)$$

So given a vector $\vec{x} = (a,b,c) \in \mathbb{R}^3$, we have

$$\begin{aligned} \vec{x} &= (a,b,c) \\ &= a(1,0,0) + b(0,1,0) + c(0,0,1) \\ &= a\vec{i} + b\vec{j} + c\vec{k} \end{aligned}$$

Notation: The following all equivalent ways to refer to a vector $\vec{x} \in \mathbb{R}^3$ with $\vec{x} = (a,b,c)$ ↑ row vector.

- $\vec{x} = \langle a,b,c \rangle$, which is called angle bracket notation
- $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, which is called a column vector
- $\vec{x} = a\vec{i} + b\vec{j} + c\vec{k}$

We will use these interchangeably.